Approximate Gumbel Last Passage Percolation

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Chapter 1

Introduction

This project formalizes the convergence properties of Last Passage Percolation (LPP) models with approximate Gumbel distributions. We establish that LPP with N-approximate Gumbel weights converges to the same GUE Tracy-Widom distribution as exact Gumbel LPP, provided that N grows appropriately with the grid size.

The main result uses a coupling argument between exact and approximate Gumbel distributions, combined with perturbation bounds to control the difference between the two models.

Chapter 2

Grid Paths and Last Passage Percolation

2.1 Basic Definitions

Definition 1 (Grid Point)**.** A *grid point* is an element of $\mathbb{N}^2$.

Definition 2 (Edge)**.** An *edge* is a pair of grid points.

Definition 3 (Up-Right Edge)**.** An edge (p, q) where $p = (x, y)$ and $q = (x', y')$ is *up-right* if either:

- $x' = x + 1$ and $y' = y$ (right step), or
- $x' = x$ and $y' = y + 1$ (up step).

Definition 4 (Grid Path)**.** A *grid path* is a list of edges.

Definition 5 (Valid Path)**.** A path is *valid* from point p to point q if it is a sequence of connected up-right edges starting at p and ending at q.

Lemma 6 (Paths Exist)**.** *For any $m, n \in \mathbb{N}$, there exists at least one valid path from $(0, 0)$ to (m, n).*

Definition 7 (LPP Value)**.** Given a weight function $w : \text{Edge} \to \mathbb{R}$ and endpoints (m, n), the *Last Passage Percolation value* is:

$$\text{LPP}_w(m, n) = \max_{\pi \in \text{Paths}(0,0;m,n)} \sum_{e \in \pi} w(e)$$

where the maximum is taken over all valid paths from $(0, 0)$ to (m, n).

Chapter 3

Gumbel and Exponential Distributions

3.1 The Gumbel Distribution

Definition 8 (Gumbel CDF)**.** The *Gumbel cumulative distribution function* is:

$$F_{\text{Gumbel}}(x) = \exp(-e^{-x})$$

Lemma 9 (Gumbel CDF is Continuous)**.** *The Gumbel CDF is continuous on* $\mathbb{R}$*.*

Definition 10 (Gumbel Grid)**.** A random field $Y : \text{Edge} \to \Omega \to \mathbb{R}$ is a *Gumbel grid* if:

1. The random variables $\{Y_e\}_{e \in \text{Edge}}$ are independent, and
2. For each edge e and $x \in \mathbb{R}$, $\mathbb{P}(Y_e \leq x) = F_{\text{Gumbel}}(x)$.

3.2 Transformation to Exponential Distribution

Lemma 11 (Gumbel Measure of Singletons is Zero)**.** *If* Y *is a Gumbel random variable, then for any* $y \in \mathbb{R}$*:*

$$\mathbb{P}(Y = y) = 0$$

Proof. This follows from the continuity of the Gumbel CDF. For any continuous CDF F, the probability of a singleton is the difference $F(y) - F(y^-)$, which is zero when F is continuous. □

Lemma 12 (Gumbel Probability of Complement)**.** *If* Y *is a Gumbel random variable, then:*

$$\mathbb{P}(Y \geq y) = 1 - F_{Gumbel}(y)$$

Lemma 13 (Gumbel to Exponential Transformation)**.** *If* Y *is a Gumbel random variable, then* $\exp(-Y)$ *has the exponential distribution with rate 1. Specifically, for* $x \geq 0$*:*

$$\mathbb{P}(\exp(-Y) \leq x) = 1 - e^{-x}$$

Proof. For $x > 0$, we have:

$$\begin{aligned}
\mathbb{P}(\exp(-Y) \leq x) &= \mathbb{P}(-Y \leq \log x) \\
&= \mathbb{P}(Y \geq -\log x) \\
&= 1 - F_{\text{Gumbel}}(-\log x) \\
&= 1 - \exp(-\exp(\log x)) \\
&= 1 - e^{-x}
\end{aligned}$$

□

Lemma 14 (Exponential Grid from Gumbel)**.** *If Y is a Gumbel grid, then $E_e = \exp(-Y_e)$ forms a grid of independent exponential random variables with rate 1.*

Proof. The independence follows from the fact that exp is a measurable function and composition with independent random variables preserves independence. The CDF property follows from the previous lemma applied to each edge. □

Chapter 4

The Coupling Construction

4.1 Approximate Gumbel Distribution

Definition 15 (Approximate Gumbel CDF)**.** For $N \geq 1$, the *N-approximate Gumbel CDF* is:

$$F_N(x) = \begin{cases} \left(1 - \frac{e^{-x}}{N}\right)^N & \text{if } x > -\log N \\ 0 & \text{otherwise} \end{cases}$$

4.2 The Coupling Function

Definition 16 (Coupling Function h_N)**.** For $N \geq 1$, define:

$$h_N(x) = -\log\left(N \cdot \left(1 - e^{-e^{-x}/N}\right)\right) - x$$

Lemma 17 (Convexity and Bounds for h_N)**.** *For $N \geq 1$, the function $h_N : \mathbb{R} \to \mathbb{R}$ satisfies:*

1. *h_N is convex on $\mathbb{R}$,*
2. *$0 < h_N(x) \leq \frac{e^{-x}}{N}$ for all $x \in \mathbb{R}$,*
3. *$\frac{e^{-x}}{3N} \leq h_N(x)$ for all $x > 0$.*

Proof. The proof uses calculus to verify convexity by showing the second derivative is non-negative. The upper bound follows from Taylor expansion of the exponential and logarithm. The lower bound for $x > 0$ uses the inequality $1 - e^{-t} \geq t - \frac{t^2}{2} + \frac{t^3}{6}$ for $t \geq 0$. □

Theorem 18 (Coupling Identity)**.** *For $N \geq 1$ and $y \in \mathbb{R}$:*

$$F_{Gumbel}(y) = F_N(h_N(y) + y)$$

Proof. Direct calculation shows both sides equal $\exp(-e^{-y})$. □

4.3 LPP Definitions

Definition 19 (Gumbel LPP)**.** For a Gumbel grid Y, define:

$$T_{\text{Gumbel}}(n) = \text{LPP}_Y(n, n)$$

Definition 20 (Approximate Gumbel LPP)**.** For a Gumbel grid Y and $N \geq 1$, define:

$$T^N_{\text{Approx}}(n) = \text{LPP}_{Y+h_N(Y)}(n, n)$$

where the weights are $w_e = Y_e + h_N(Y_e)$ for each edge e.

Definition 21 (Exponential LPP)**.** For a grid E of exponential random variables:

$$L_{\text{Exp}}(n) = \text{LPP}_E(n, n)$$

Chapter 5

Perturbation Analysis

Lemma 22 (Perturbation Bounds)**.** *Let* Π *be a finite nonempty set, and* $S_A, S_B : \Pi \to \mathbb{R}$ *be functions. Suppose* π^* *maximizes* S_A*. Define:*

- $M_A = S_A(\pi^*)$
- $M_{A+B} = \max_{\pi\in\Pi}(S_A(\pi) + S_B(\pi))$
- $m_B = \min_{\pi\in\Pi} S_B(\pi)$
- $M_B = \max_{\pi\in\Pi} S_B(\pi)$

Then:

$$m_B \leq S_B(\pi^*) \leq M_{A+B} - M_A \leq M_B$$

Proof. The first inequality is immediate. For the second, note that $M_{A+B} \geq S_A(\pi^*) + S_B(\pi^*) = M_A + S_B(\pi^*)$. For the third, let π' maximize $S_A + S_B$. Then:

$$M_{A+B} - M_A = S_A(\pi') + S_B(\pi') - S_A(\pi^*) \leq S_B(\pi') \leq M_B$$

where we used $S_A(\pi^*) \geq S_A(\pi')$. □

5.1 Coupling Bounds

Theorem 23 (Coupling Upper Bound)**.** *For* $N \geq 1$*, a Gumbel grid* Y*, and* $n \in \mathbb{N}$*:*

$$T^N_{Approx}(n) - T_{Gumbel}(n) \leq \frac{1}{N} \cdot L_{Exp}(n)$$

where $L_{Exp}(n)$ *is computed with weights* $E_e = e^{-Y_e}$*.*

Proof. Let π^* be the maximizing path for T_{Gumbel}. By Lemma 17, for each edge e:

$$h_N(Y_e) \leq \frac{e^{-Y_e}}{N}$$

Summing over the maximizing path for T^N_{Approx} and using Lemma 22 gives the result. □

Theorem 24 (Coupling Lower Bound)**.** *For $N \geq 1$, a Gumbel grid Y, and $n \in \mathbb{N}$:*

$$2n \cdot h_N \left(\frac{T_{Gumbel}(n)}{2n} \right) \leq T^N_{Approx}(n) - T_{Gumbel}(n)$$

Proof. Let π^* be the geodesic for T_{Gumbel}, which has length $2n$. By Jensen's inequality applied to the convex function h_N:

$$\frac{1}{2n} \sum_{e \in \pi^*} h_N(Y_e) \geq h_N \left(\frac{1}{2n} \sum_{e \in \pi^*} Y_e \right) = h_N \left(\frac{T_{\text{Gumbel}}(n)}{2n} \right)$$

The result follows since $T^N_{\text{Approx}}(n) \geq \sum_{e \in \pi^*} (Y_e + h_N(Y_e))$. □

Chapter 6

Convergence Properties

6.1 Convergence Definitions

Definition 25 (Convergence in Probability to Zero)**.** A sequence of random variables $\{X_n\}$ *converges in probability to zero* if:

$$\forall \varepsilon > 0, \quad \mathbb{P}(|X_n| > \varepsilon) \to 0 \text{ as } n \to \infty$$

Definition 26 (Convergence in Probability to a Constant)**.** A sequence of random variables $\{X_n\}$ *converges in probability to c* if:

$$\forall \varepsilon > 0, \quad \mathbb{P}(|X_n - c| > \varepsilon) \to 0 \text{ as } n \to \infty$$

6.2 Known Results (Axiomatized)

The following properties capture known results from the literature that we assume as axioms:

Definition 27 (Exact Gumbel Convergence Property)**.** For a Gumbel grid and appropriate constants $C_g, \sigma_g > 0$:

$$\frac{T_{\text{Gumbel}}(n) - C_g \cdot n}{\sigma_g \cdot n^{1/3}} \xrightarrow{d} F_{\text{GUE}}$$

where F_{GUE} is the GUE Tracy-Widom distribution.

Definition 28 (Time Constant for Gumbel LPP)**.** There exists a constant $D_\ell > 0$ such that:

$$\frac{T_{\text{Gumbel}}(n)}{n} \xrightarrow{\mathbb{P}} D_\ell$$

Definition 29 (Time Constant for Exponential LPP)**.** There exists a constant $D_L > 0$ such that:

$$\frac{L_{\text{Exp}}(n)}{n} \xrightarrow{\mathbb{P}} D_L$$

6.3 Slutsky's Theorem

Theorem 30 (Slutsky Upper Bound)**.** *For random variables X, Y and constants r, ε:*

$$\mathbb{P}(X + Y \leq r) \leq \mathbb{P}(X \leq r + \varepsilon) + \mathbb{P}(|Y| > \varepsilon)$$

Theorem 31 (Slutsky Lower Bound)**.** *For random variables X, Y and constants r, ε:*

$$\mathbb{P}(X \leq r - \varepsilon) \leq \mathbb{P}(X + Y \leq r) + \mathbb{P}(|Y| > \varepsilon)$$

Theorem 32 (Slutsky's Theorem for CDFs)**.** *Suppose $X_n \xrightarrow{d} F$ (convergence in distribution to a continuous CDF F) and $Y_n \xrightarrow{\mathbb{P}} 0$. Then:*

$$X_n + Y_n \xrightarrow{d} F$$

Proof. Fix $r \in \mathbb{R}$ and $\varepsilon > 0$. By Theorems 30 and 31:

$$\begin{aligned} \mathbb{P}(X_n \leq r - \varepsilon) - \mathbb{P}(|Y_n| > \varepsilon) &\leq \mathbb{P}(X_n + Y_n \leq r) \\ &\leq \mathbb{P}(X_n \leq r + \varepsilon) + \mathbb{P}(|Y_n| > \varepsilon) \end{aligned}$$

Taking limits and using continuity of F gives the result. □

6.4 Auxiliary Convergence Lemmas

Lemma 33 (Product Convergence)**.** *If $Y_n \xrightarrow{\mathbb{P}} c$ and $a_n \to 0$, then $a_n \cdot Y_n \xrightarrow{\mathbb{P}} 0$.*

Lemma 34 (Deterministic Factor Limit)**.** *For $\alpha > 2/3$:*

$$\frac{n}{\lfloor n^\alpha \rfloor \cdot n^{1/3}} \to 0 \text{ as } n \to \infty$$

Proof. We have:

$$\frac{n}{\lfloor n^\alpha \rfloor \cdot n^{1/3}} \leq \frac{2n}{n^\alpha \cdot n^{1/3}} = 2n^{2/3-\alpha}$$

Since $\alpha > 2/3$, the exponent is negative and the limit is zero. □

Chapter 7

Main Theorem

Theorem 35 (Approximate Gumbel Convergence: Critical Threshold at $\alpha = 2/3$)**.** *Assume the properties in Definitions 27, 28, and 29. Let $N_n = \lfloor n^\alpha \rfloor$ for some $\alpha > 0$. For a sequence of Gumbel grids $Y^{(n)}$:*

1. ***(Convergence for** $\alpha > 2/3$**)** If $\alpha > 2/3$, then for any $r \in \mathbb{R}$:*

$$\mathbb{P}\left(\frac{T^{N_n}_{Approx}(n) - C_g \cdot n}{\sigma_g \cdot n^{1/3}} \leq r\right) \to F_{GUE}(r)$$

as $n \to \infty$. That is, the approximate Gumbel LPP converges to the same GUE Tracy-Widom distribution as the exact Gumbel LPP.

2. ***(Divergence for** $\alpha < 2/3$**)** If $\alpha < 2/3$, then the fluctuations diverge:*

$$\frac{T^{N_n}_{Approx}(n) - C_g \cdot n}{\sigma_g \cdot n^{1/3}} \xrightarrow{\mathbb{P}} +\infty$$

as $n \to \infty$. More precisely, for any $M > 0$:

$$\mathbb{P}\left(\frac{T^{N_n}_{Approx}(n) - C_g \cdot n}{\sigma_g \cdot n^{1/3}} > M\right) \to 1$$

Thus $\alpha = 2/3$ represents a sharp threshold: the approximation parameter N must grow faster than $n^{2/3}$ for the limiting distribution to remain Tracy-Widom GUE.

Proof.

By Theorems 23 and 24:

$$2n \cdot h_{N_n}\left(\frac{T_{\text{Gumbel}}(n)}{2n}\right) \leq T^{N_n}_{\text{Approx}}(n) - T_{\text{Gumbel}}(n) \leq \frac{1}{N_n} L_{\text{Exp}}(n)$$

Dividing by $\sigma_g n^{1/3}$:

$$\frac{2n \cdot h_{N_n}(T_{\text{Gumbel}}(n)/(2n))}{\sigma_g n^{1/3}} \leq \frac{T^{N_n}_{\text{Approx}}(n) - T_{\text{Gumbel}}(n)}{\sigma_g n^{1/3}} \leq \frac{L_{\text{Exp}}(n)}{N_n \sigma_g n^{1/3}}$$

Case 1: $\alpha > 2/3$**.**
By Definition 29, $L_{\text{Exp}}(n)/n \xrightarrow{\mathbb{P}} D_L$. We have:

$$\frac{L_{\text{Exp}}(n)}{N_n \sigma_g n^{1/3}} = \frac{L_{\text{Exp}}(n)}{n} \cdot \frac{n}{N_n \sigma_g n^{1/3}}$$

By Lemma 34, $n/(N_n n^{1/3}) \to 0$ when $\alpha > 2/3$. Therefore by Lemma 33:

$$\frac{T_{\text{Approx}}^{N_n}(n) - T_{\text{Gumbel}}(n)}{\sigma_g n^{1/3}} \xrightarrow{\mathbb{P}} 0$$

By Definition 27:

$$\frac{T_{\text{Gumbel}}(n) - C_g n}{\sigma_g n^{1/3}} \xrightarrow{d} F_{\text{GUE}}$$

Applying Slutsky's Theorem (Theorem 32) gives:

$$\frac{T_{\text{Approx}}^{N_n}(n) - C_g n}{\sigma_g n^{1/3}} \xrightarrow{d} F_{\text{GUE}}$$

Case 2: $\alpha < 2/3$**.**
By the lower bound from Theorem 24 and Lemma 17, for $x > 0$:

$$h_{N_n}(x) \geq \frac{e^{-x}}{3N_n}$$

By Definition 28, $T_{\text{Gumbel}}(n)/n \xrightarrow{\mathbb{P}} D_\ell$ where $D_\ell > 0$. Therefore $T_{\text{Gumbel}}(n)/(2n) \xrightarrow{\mathbb{P}} D_\ell/2 > 0$, which means for large n, $T_{\text{Gumbel}}(n)/(2n)$ is bounded away from zero with high probability. Thus:

$$\frac{2n \cdot h_{N_n}(T_{\text{Gumbel}}(n)/(2n))}{\sigma_g n^{1/3}} \geq \frac{2n \cdot e^{-T_{\text{Gumbel}}(n)/(2n)}}{3N_n \sigma_g n^{1/3}} \geq \frac{C \cdot n}{N_n n^{1/3}} = C \cdot n^{2/3-\alpha}$$

for some constant $C > 0$ (with high probability). When $\alpha < 2/3$, the exponent $2/3 - \alpha > 0$, so this lower bound diverges to $+\infty$ as $n \to \infty$. Therefore:

$$\frac{T_{\text{Approx}}^{N_n}(n) - T_{\text{Gumbel}}(n)}{\sigma_g n^{1/3}} \xrightarrow{\mathbb{P}} +\infty$$

Since the scaled $T_{\text{Gumbel}}(n)$ converges in distribution (hence is tight), the scaled $T_{\text{Approx}}^{N_n}(n)$ must diverge to $+\infty$ in probability. □