An F-KPP Strong Selection Principle

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Abstract

We consider a diffusion-branching-selection particle system introduced by Groisman, Jonckheere, and Martínez in [3]. Using the techniques of [6] and [1], we prove for this system a strong selection principle: as $N \to \infty$, the stationary empirical measure of the $N-$particle system converges to the minimal speed traveling wave profile of the Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation.

1 Introduction and Main Result

We first recall the dynamics of the system introduced in [3]. Let $\xi = (\xi(1), ..., \xi(N)) \in \mathbb{R}^N$ be a system of N particles. Define the jump operator which moves the leftmost of a pair of particles to the position of the rightmost

$$\theta_{ij}(\xi)(k) = \begin{cases} \max(\xi(i), \xi(j)) & k \in \{i, j\} \\ \xi(k) & k \notin \{i, j\}. \end{cases}$$

The $N-$particle process ξ_t has generator

$$\mathcal{L}_N f(\xi) = \frac{1}{2} \sum_{i=1}^{N} \partial_{ii} f(\xi) + \frac{1}{2(N-1)} \sum_{i=1}^{N} \sum_{j \neq i} \big(f(\theta_{ij}\xi) - f(\xi)\big).$$

Equivalently, at rate $N/2$, an unordered pair of particle is chosen uniformly at random and the leftmost particle in the pair jumps on top of the rightmost. Let $\xi_{[i]}$ be the ith particle from the left. Let $\eta_t(i) = \xi_{[i+1]} - \xi_{[1]}$ be the process seen from its minimum.

In [3], the authors prove the existence and uniqueness of a stationary distribution ν_N for the η_t. Let $\overline{\nu}_N$ be the distribution on configurations with the leftmost particle placed at the origin and the remaining particles distributed as ν_N.

Our aim is to follow the strategy introduced in [6] and later used in [1] to prove a strong selection principle for this system. More precisely, we aim to show that after centering at the empirical median, the stationary empirical measure converges (as $N \to \infty$) to the minimal-speed traveling wave profile for the Fisher-KPP equation. This strategy is as follows.

1. We prove tightness of the laws of the median-centered stationary empirical measures.
2. We prove that every subsequential limit is an invariant law for the median-centered F-KPP flow.
3. We combine the stretching property of the F-KPP equation with a microscopic speed identity to identify the invariant law.

Our main tool will be the speed functional

$$\mathcal{S}(\mu) := \int_{\mathbb{R}} F_\mu(x)(1 - F_\mu(x))dx \tag{1}$$

where μ is a probability measure on $\mathbb{R}$ and F_μ its cumulative distribution function. In fact, properties of this functional will be very nearly the only inputs in steps (1) and (3).

Let $U_{\min}$ be the centered minimal-speed traveling-wave profile for

$$\partial_t u = \frac{1}{2}\partial_{xx}u + u(u-1) \tag{2}$$

i.e. one such that

$$u(x,t) = U_{\min}(x - \sqrt{2}t)$$

solves Equation (2) centered such that $U_{\min}(0) = \frac{1}{2}$. Let $\pi_{\min}$ be the probability measure whose cumulative distribution function is $U_{\min}$.

For a Polish space S, write $\mathcal{P}(S)$ for the space of probability measures on S endowed with the weak topology (which is itself a Polish space). Write $\mu_N \implies \mu$ if μ_N converges to μ in this space. If ψ is a random measure, i.e. a random variable taking values in $\mathcal{P}(\mathbb{R})$, write $\mathcal{L}(\psi)$ for its law, an element of $\mathcal{P}(\mathcal{P}(\mathbb{R}))$.

Let $(\xi_i^N(t))_{1\le i\le N}$ be the $N-$particle process started from $\overline{\nu}_N$. Define the empirical measure $\mu_N(t)$ and empirical CDF $F_N(x,t)$ by

$$\mu_N(t) := \frac{1}{N}\sum_{i=1}^N \delta_{\xi_i^N(t)}, \quad F_N(x,t) = \mu_N(t)((-\infty,x]).$$

For a probability measure μ, write

$$a(\mu) := \inf\{x \in \mathbb{R} : \mu((-\infty,x]) \ge \frac{1}{2}\}$$

then write

$$M_N(t) = a(\mu_N(t))$$

for the lower median process of the empirical measure. Define median-centered empirical measures are

$$\psi_N(t) := \frac{1}{N}\sum_{i=1}^N \delta_{\xi_i^N(t) - M_N(t)}, \quad U_N(x,t) = F_N(M_N(t) + x, t).$$

Since the process seen from its minimum is stationary and median centering is translation invariant, the law $\zeta_N := \mathcal{L}(\psi_N(t)) \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ does not depend on $t \ge 0$.

Theorem 1. *As $N \to \infty$, $\zeta_N \implies \delta_{\pi_{\min}} \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$. Equivalently, for each fixed $t \ge 0$,*

$$\sup_{x\in\mathbb{R}} |U_N(x,t) - U_{\min}(x)| \to 0$$

in probability.

1.1 AI Disclosure

GPT-5.6 Sol was used for literature review and proofreading. GPT-5.6 Pro produced and proved the estimates Equation (9) and Equation (46) and several of the parts of the proof of Theorem 2.

2 Preliminary Properties

In this section we record some basic facts about the speed functional, the particle system, and the F-KPP equation.

Proposition 1 (Basic Properties of the Speed Functional)**.** *Let X and Y be independent random variables with law $\mu \in \mathcal{P}(\mathbb{R})$.*

1. $\mathcal{S}(\mu) = \frac{1}{2}\mathbb{E}|X - Y|$ *with $\mathcal{S}$ defined in Equation* (1)*.*
2. *The map $\mathcal{S} : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ is lower semicontinous.*
3. *If $a = a(\mu)$, then*
$$\mathcal{S}(\mu) \leq \int_{\mathbb{R}} |x - a|\mu(dx) \leq 2\mathcal{S}(\mu).$$
Thus $\mathcal{S}(\mu) < \infty$ if and only if μ has a finite first moment.

Proof. 1. Note that

$$\mathbb{E}|X - Y| = \int_{\mathbb{R}} \mathbb{P}(X \leq z < Y) + \mathbb{P}(Y \leq z < X)dz = 2\int_{\mathbb{R}} F_\mu(z)(1 - F_\mu(z))dz.$$

2. Note that the function $(x, y) \mapsto |x - y|$ is lower semicontinuous and bounded below. Suppose $\mu_N \Longrightarrow \mu \in \mathcal{P}(\mathbb{R})$. Then $\mu_N \otimes \mu_N \Longrightarrow \mu \otimes \mu$. Then Portmanteau theorem gives

$$\liminf \mathcal{S}(\mu_N) = \frac{1}{2} \liminf \int_{\mathbb{R}^2} |x - y|\mu_N(dx)\mu_N(dy) \geq \frac{1}{2} \int_{\mathbb{R}^2} |x - y|\mu(dx)\mu(dy) = \mathcal{S}(\mu).$$

3. The triangle inequality gives

$$|X - Y| \leq |X - a| + |Y - a|$$

so taking expectations and using (1) gives

$$\mathcal{S}(\mu) \leq \mathbb{E}|X - a|.$$

Now

$$\mathbb{E}|X - a| = \int_{-\infty}^{a} F_\mu(x)dx + \int_{a}^{\infty} (1 - F_\mu(x))dx.$$

Now $F_\mu(x) \leq 1/2$ for $x < a$ and $F_\mu(x) \geq 1/2$ for $x > a$ therefore for $x < a$ we have

$$F_\mu(x)(1 - F_\mu(x)) \geq \frac{1}{2}F_\mu(x)$$

and for $x > a$ we have

$$F_\mu(x)(1 - F_\mu(x)) \geq \frac{1}{2}(1 - F_\mu(x)).$$

Combining these facts completes the proof. □

Now we turn our attention to the particle system and state a somewhat modified version of the hydrodynamic limit from [3].

First we need some notation. Let $\mu \in \mathcal{P}(\mathbb{R})$ have CDF $F_\mu(x)$. Define the uncentered F-KPP map $\Psi_t(\mu) \in \mathcal{P}(\mathbb{R})$ to be the measure with CDF given by $F_\mu(t, x)$ such that $F_\mu(t, x)$ satisfies Equation (2) and $F_\mu(0, x) = F_\mu(x)$. Define the centered flow

$$\Phi_t(\mu) = \tau_{-a(\Psi_t(\mu))}\Psi_t(\mu)$$

where τ_x is the right translation map and $a(\nu)$ is the lower median of the measure ν.

Remark 1. The well-definition of $\Psi_t(\mu)$ (in particular that the F-KPP equation evolves CDFs to CDFs) follows from the proof of Proposition 2.

Theorem 2 (Hydrodynamic limit). *Let $\mu_N(t)$ be the empirical distribution of the $N-$particle system with any initial data. Suppose $\mathcal{L}(\mu_N(0)) \Longrightarrow \Lambda \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$. Let μ be a random variable taking values in $\mathcal{P}(\mathbb{R})$ with law Λ. Then for all $t \geq 0$,*

$$\mathcal{L}(\mu_N(0), \mu_N(t)) \Longrightarrow \mathcal{L}(\mu, \Psi_t(\mu)).$$

This result is only a mild upgrade of the original result in [3] and uses very few new ideas. The proof is provided in the appendix.

Now we turn to basic properties of the F-KPP equation. The proofs of these properties are straightforward and are therefore deferred to an appendix.

Proposition 2 (Regularity for the F-KPP flow). *For every $t > 0$ and $\mu \in \mathcal{P}(\mathbb{R})$, the measure $\Psi_t\mu$ has continuous and strictly increasing CDF and consequently a unique median.*

This proposition is useful because of the following elementary fact: the lower-median-centering map $\nu \mapsto \tau_{-a(\nu)}\nu$ is continuous at measures ν with continuous and strictly increasing CDF.

We will state some properties of the F-KPP equation using the notion of steepness. Let U and V be distributions functions on $\mathbb{R}$. We say that U is steeper than V and write $U \geq_{stp} V$ if for any $c \in \mathbb{R}$ and $x_1 \leq x_2$,

$$U(x_1) > V(x_1 + c) \implies U(x_2) \geq V(x_2 + c).$$

We say that a measure μ is steeper than a measure ν and write $\mu \geq_{stp} \nu$ if the CDFs satisfy $F_\mu \geq_{stp} F_\nu$.

Proposition 3 (Closure of steepness). *Suppose $\mu_N \Longrightarrow \mu$ and $\nu_N \Longrightarrow \nu$ where the CDFs F_μ and F_ν are continuous and $\mu_N \geq_{stp} \nu_N$ for all N. Then $\mu \geq_{stp} \nu$.*

Proposition 4 (F-KPP preserves steepness ordering). *Suppose $\mu \geq_{stp} \nu$. Then for every $t \geq 0$, $\Psi_t\mu \geq_{stp} \Psi_t\nu$ and $\Phi_t\mu \geq_{stp} \Phi_t\nu$.*

Bramson's memoir [2] uses a slightly different notion of steepness that is not preserved under pointwise limits. The above definition is from [1]. However, arguments from Bramson's work imply Proposition 4.

Proposition 5. *We have*

$$\mathcal{S}(\pi_{\min}) = \sqrt{2}. \tag{3}$$

If μ has continuous strictly increasing CDF and the median $a(\mu) = 0$ and $\pi_{\min} \geq_{stp} \mu$, then

$$\mathcal{S}(\mu) \geq \sqrt{2} \tag{4}$$

with equality if and only if $\mu = \pi_{\min}$.

Proposition 6. *Let $H(x) = \mathbb{1}_{x \geq 0}$. Let λ be the measure with CDF H. Then*

$$\Phi_t\lambda \Longrightarrow \pi_{\min}$$

as $t \to \infty$.

In other words, solutions to the F-KPP equation with step initial data converge in shape to the minimal speed traveling wave. This follows from Theorems 13 and 17 in the original KPP paper [4].

3 Proof of the Main Result

Lemma 1 (Tightness)**.** *For every $N \geq 2$,*

$$\int_{\mathcal{P}(\mathbb{R})} \mathcal{S}(\mu)\zeta_N(d\mu) = \frac{N-1}{N} v_N \leq \sqrt{2}$$

so

$$\int_{\mathcal{P}(\mathbb{R})} \int_{\mathbb{R}} |x|\mu(dx)\zeta_N(d\mu) \leq 2\sqrt{2}$$

where v_N is the asymptotic velocity of the $N-$particle cloud and therefore $(\zeta_N)_{N\geq 2}$ is tight in $\mathcal{P}(\mathcal{P}(\mathbb{R})$.

Proof. This follows from Proposition 1 and Equation (4.5) in [3]. □

Lemma 2 (Subsequential limits are fixed points)**.** *Let ζ be a subsequential limit of $(\zeta_N)_{N\geq 2}$, i.e. we have a subsequence ζ_{N_j} such that $\zeta_{N_j} \Longrightarrow \zeta$. Then for all $t \geq 0$,*

$$\zeta = (\Phi_t)_\# \zeta.$$

Proof. Let $\mu_{N_j}(t)$ be the empirical distribution of the process where $\mu_{N_j}(0)$ has gaps distributed according to the stationary law and $a(\mu_{N_j}(0)) = 0$. Let μ be a random measure with law ζ. By Theorem 2, for fixed $t > 0$, we have as $j \to \infty$,

$$\mathcal{L}(\mu_{N_j}(0), \mu_{N_j}(t)) \Longrightarrow \mathcal{L}(\mu, \Psi_t \mu).$$

Since $\Psi_t\mu$ has continuous and strictly increasing CDF, the median centering map $\alpha : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ defined by $\alpha(\mu) = \tau_{-a(\mu)}\mu$ is continuous at $\Psi_t\mu$. Therefore

$$\psi_N(t) = \alpha(\mu_N(t)) \Longrightarrow \alpha(\Psi_t(\mu)) = \Phi_t(\mu).$$

On the other hand, we know that $\mathcal{L}(\psi_N(t)) = \zeta_N \Longrightarrow \zeta$. Therefore $\Phi_t(\mu)$ is equal to ζ in law, i.e. $(\Phi_t)_\#\zeta = \zeta$. □

Proof of Theorem 1. By Lemma 1, every subsequence has a further subsequence $\zeta_{N_k} \Longrightarrow \zeta \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$. By Skorokhod's representation theorem, we can choose μ be a random measure with law ζ and μ_{N_k} random measures with laws ζ_{N_k} such that $\mu_{N_k} \to \mu$ weakly almost surely. Then since $\mathcal{S}$ is lower semicontinuous and by Lemma 1 and Fatou's lemma, we have

$$\mathbb{E}(\mathcal{S}(\mu)) \leq \mathbb{E}(\liminf \mathcal{S}(\mu_{N_k})) \leq \liminf \mathbb{E}(\mathcal{S}(\mu_{N_k})) \leq \sqrt{2}. \tag{5}$$

On the other hand,

$$\lambda \geq_{stp} \mu.$$

Therefore by Proposition 4 for all t,

$$\Phi_t\lambda \geq_{stp} \Phi_t\mu$$

almost surely. But $\Phi_t\mu$ is equal to μ in law, therefore if ν is $\zeta-$sampled, for any t, $\Phi_t\lambda \geq_{stp} \nu$, $\zeta-$almost surely. We can choose ν such that $\Phi_n\lambda \geq_{stp} \nu$ for all $n \in \mathbb{N}$ almost surely by sampling from the countable intersection of full-measure events. Moreover, we know that ζ samples have continuous and strictly increasing CDF almost surely by Proposition 2. Now by Proposition 6 and Proposition 3, we know that

$$\pi_{\min} = \lim_{n\to\infty} \Phi_n\lambda \geq_{stp} \nu$$

almost surely. Hence Proposition 5 means that $\mathcal{S}(\nu) \geq \sqrt{2}$ almost surely. This combined with Equation (5) means that $S(\nu) = \sqrt{2}$ almost surely.

Therefore by Proposition 5, $\nu = \pi_{\min}$ almost surely. Thus $\zeta = \delta_{\pi_{\min}}$. This is to say that any subsequence of the ζ_N has a further subsequence which converges to $\delta_{\pi_{\min}}$, so $\zeta_N \Longrightarrow \delta_{\pi_{\min}}$ as desired.

□

A Proof of the Hydrodynamic Limit

Lemma 3. *Suppose that* $v, w : \mathbb{R} \times [0, T] \to [0, 1]$ *are bounded mild solutions of*

$$\partial_t v = \frac{1}{2}\partial_{xx} v + f(v) + r, \qquad \partial_t w = \frac{1}{2}\partial_{xx} w + f(w), \tag{6}$$

where

$$\sup_{x\in\mathbb{R},\ 0\leq s\leq T} |r(x,s)| \leq \eta. \tag{7}$$

If

$$v(x,0) - w(x,0) \leq a \qquad (x \in \mathbb{R}) \tag{8}$$

for some $a \geq 0$*, then*

$$v(x,t) - w(x,t) \leq ae^t + \eta(e^t - 1) \qquad (x \in \mathbb{R},\ 0 \leq t \leq T). \tag{9}$$

Proof. Let $z = v - w$. Since

$$f(v) - f(w) = (v + w - 1)(v - w),$$

we have

$$\partial_t z = \frac{1}{2}\partial_{xx} z + c(x,t)z + r(x,t), \qquad c(x,t) := v(x,t) + w(x,t) - 1, \tag{10}$$

where $-1 \leq c \leq 1$. Define

$$B(t) := ae^t + \eta(e^t - 1).$$

Then $B' = B + \eta$, and therefore

$$\begin{aligned} \partial_t(z - B) - \frac{1}{2}\partial_{xx}(z - B) - c(z - B) &= r - B' + cB \\ &= r - \eta + (c-1)B \leq 0. \end{aligned} \tag{11}$$

Since $z(\cdot, 0) - B(0) \leq 0$, the comparison principle for bounded mild solutions gives $z \leq B$, proving Equation (9). Applying the same argument to $w - v$ gives the corresponding lower bound. In particular, if $r \equiv 0$ and

$$\|v(\cdot,0) - w(\cdot,0)\|_\infty \leq a,$$

then

$$\|v(\cdot,t) - w(\cdot,t)\|_\infty \leq ae^t. \tag{12}$$

□

Lemma 4. *The F-KPP flow is continuous as a map* $\mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$*. Equivalently by Proposition 2,*

$$\rho_n \Longrightarrow \rho \quad \Longrightarrow \quad \|u^{\rho_n}(\cdot,t) - u^{\rho}(\cdot,t)\|_\infty \longrightarrow 0. \tag{13}$$

Proof. Suppose that $\rho_n \Longrightarrow \rho$. For every $\delta > 0$, the characterization of weak convergence by the Lévy–Prokhorov metric gives, for all sufficiently large n,

$$F_\rho(x-\delta) - \delta \le F_{\rho_n}(x) \le F_\rho(x+\delta) + \delta \qquad (x \in \mathbb{R}). \tag{14}$$

Since Equation (2) is invariant under spatial translations, the function $x \mapsto u^\rho(x+\delta, t)$ is the solution with initial datum $x \mapsto F_\rho(x+\delta)$. Applying Equation (9) with $r = 0$ to the upper inequality in Equation (14) gives

$$u^{\rho_n}(x,t) \le u^\rho(x+\delta, t) + \delta e^t. \tag{15}$$

Similarly, the lower inequality in Equation (14) gives

$$u^{\rho_n}(x,t) \ge u^\rho(x-\delta, t) - \delta e^t. \tag{16}$$

Define

$$\omega_{\rho,t}(\delta) := \sup_{x\in\mathbb{R}} \big(u^\rho(x+\delta,t) - u^\rho(x,t) \big).$$

Combining Equation (15) and Equation (16) yields

$$\| u^{\rho_n}(\cdot,t) - u^\rho(\cdot,t) \|_\infty \le \omega_{\rho,t}(\delta) + \delta e^t \tag{17}$$

for all sufficiently large n. Since $u^\rho(\cdot,t)$ is a continuous distribution function, it is uniformly continuous on $\mathbb{R}$, and hence

$$\omega_{\rho,t}(\delta) \longrightarrow 0 \qquad \text{as } \delta \downarrow 0.$$

Taking first $n \to \infty$ and then $\delta \downarrow 0$ in Equation (17) proves Equation (13). $\square$

We shall also use the following elementary fact. Let G_N be random distribution functions and let G be a deterministic continuous distribution function. If

$$G_N(x) \longrightarrow G(x) \qquad \text{in probability} \tag{18}$$

for every $x \in \mathbb{R}$, then

$$\sup_{x\in\mathbb{R}} |G_N(x) - G(x)| \longrightarrow 0 \qquad \text{in probability.} \tag{19}$$

Proof of Theorem 2. Throughout the proof, let

$$E := \mathcal{P}(\mathbb{R}).$$

For $\rho \in E$, write

$$u^\rho(x,t) := F_{\Psi_t \rho}(x).$$

By Proposition 2, $u^\rho(\cdot,t)$ is a continuous, strictly increasing distribution function for every $t > 0$.

Our first step is to generalize the result in [3] to apply to weakly convergent initial data. Here the limiting distribution might be discontinuous, so we make a weaker assumption than in the original result.

For each N, let

$$\xi^{N,0} = \big(\xi^{N,0}(1), \dots, \xi^{N,0}(N)\big)$$

be a deterministic initial configuration, and let

$$\rho_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi^{N,0}(i)}.$$

Assume that

$$\rho_N \Longrightarrow \rho. \tag{20}$$

We claim that, for every fixed $t > 0$,

$$\sup_{x\in\mathbb{R}} |F_N(x,t) - u^\rho(x,t)| \longrightarrow 0 \qquad \text{in probability.} \tag{21}$$

Let $\mathbb{P}_{\xi^{N,0}}$ and $\mathbb{E}_{\xi^{N,0}}$ denote probability and expectation for the particle system started from $\xi^{N,0}$, and define

$$u_N(x,s) := \mathbb{E}_{\xi^{N,0}}[F_N(x,s)].$$

Then

$$u_N(x,0) = F_{\rho_N}(x).$$

The computation in the proof of the hydrodynamic limit in [3] gives, in the bounded mild sense,

$$\partial_s u_N(x,s) = \frac{1}{2}\partial_{xx} u_N(x,s) - \frac{N}{N-1}\mathbb{E}_{\xi^{N,0}}\left[F_N(x,s)(1-F_N(x,s))\right]. \tag{22}$$

Since

$$\mathbb{E}_{\xi^{N,0}}\left[F_N(x,s)(1-F_N(x,s))\right] = u_N(x,s)(1-u_N(x,s)) - \mathrm{Var}_{\xi^{N,0}}(F_N(x,s)), \tag{23}$$

equation Equation (22) becomes

$$\partial_s u_N = \frac{1}{2}\partial_{xx} u_N + f(u_N) + r_N, \tag{24}$$

where

$$r_N(x,s) := \frac{N}{N-1}\,\mathrm{Var}_{\xi^{N,0}}(F_N(x,s)) - \frac{1}{N-1}u_N(x,s)(1-u_N(x,s)). \tag{25}$$

For $1 \le i \le N$, set

$$I_i(x,s) := \mathbf{1}_{\{\xi_i^N(s)\le x\}}.$$

By [3, Lemma 3.2], uniformly over deterministic initial configurations,

$$\left|\sum_{i,j=1}^{N} \mathrm{Cov}_{\xi^{N,0}}\big(I_i(x,s), I_j(y,s)\big)\right| \le 2Ne^s. \tag{26}$$

Taking $x = y$ in Equation (26) gives

$$\sup_{\xi^{N,0}}\sup_{x\in\mathbb{R}} \mathrm{Var}_{\xi^{N,0}}(F_N(x,s)) \le \frac{2e^s}{N}. \tag{27}$$

Consequently, for $0 \le s \le T$,

$$\sup_{x\in\mathbb{R},\ 0\le s\le T} |r_N(x,s)| \le \frac{2e^T + \frac{1}{4}}{N-1} =: \varepsilon_{N,T}, \tag{28}$$

and

$$\varepsilon_{N,T} \longrightarrow 0. \tag{29}$$

Let

$$v_N(x,s) := u^{\rho_N}(x,s).$$

The functions u_N and v_N have the same initial datum:

$$u_N(x,0) = v_N(x,0) = F_{\rho_N}(x).$$

Applying Equation (9) in both directions to Equation (24) and Equation (2), with $a = 0$ and $\eta = \varepsilon_{N,T}$, gives

$$\|u_N(\cdot, s) - v_N(\cdot, s)\|_\infty \leq \varepsilon_{N,T}(e^s - 1) \qquad (0 \leq s \leq T). \tag{30}$$

Thus, for each fixed $t > 0$,

$$\|u_N(\cdot, t) - v_N(\cdot, t)\|_\infty \longrightarrow 0. \tag{31}$$

On the other hand, Equation (20) and Equation (13) give

$$\|v_N(\cdot, t) - u^\rho(\cdot, t)\|_\infty \longrightarrow 0. \tag{32}$$

Combining Equation (31) and Equation (32), we obtain

$$\|u_N(\cdot, t) - u^\rho(\cdot, t)\|_\infty \longrightarrow 0. \tag{33}$$

It remains to remove the expectation. For every fixed $x \in \mathbb{R}$ and $\delta > 0$, Chebyshev's inequality and Equation (27) give

$$\mathbb{P}_{\xi^{N,0}}\left(|F_N(x, t) - u_N(x, t)| > \delta\right) \leq \frac{2e^t}{N\delta^2}. \tag{34}$$

Equation (33) and Equation (34) imply that

$$F_N(x, t) \longrightarrow u^\rho(x, t) \qquad \text{in probability} \tag{35}$$

for every fixed $x \in \mathbb{R}$. Since $u^\rho(\cdot, t)$ is a continuous distribution function, Equation (19) yields Equation (21). Equivalently, whenever Equation (20) holds,

$$\mu_N(t) \longrightarrow \Psi_t\rho \qquad \text{in probability in } E. \tag{36}$$

Now we proceed with our final upgrade: we allow for random initial empirical measures.

Let

$$\Lambda_N := \mathcal{L}(\mu_N(0)).$$

By assumption,

$$\Lambda_N \Longrightarrow \Lambda \qquad \text{in } \mathcal{P}(E). \tag{37}$$

For each N, let

$$E_N := \left\{ \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} : x_1, \ldots, x_N \in \mathbb{R} \right\}.$$

Since $\mu_N(0)$ is an empirical measure of N particles,

$$\Lambda_N(E_N) = 1.$$

Because E is Polish, the Skorokhod representation theorem applied to Equation (37) gives E-valued random variables R_N and R, defined on a common probability space, such that

$$R_N \sim \Lambda_N, \qquad R \sim \Lambda, \tag{38}$$

and

$$R_N \longrightarrow R \qquad \text{almost surely in } E. \tag{39}$$

Moreover, $R_N \in E_N$ almost surely.

For $\rho \in E_N$, define its canonical ordered representative by

$$q_{N,k}(\rho) := \inf\left\{x \in \mathbb{R} : F_\rho(x) \geq \frac{k}{N}\right\}, \qquad 1 \leq k \leq N,$$

and set

$$\mathbf{q}_N(\rho) := \big(q_{N,1}(\rho), \ldots, q_{N,N}(\rho)\big).$$

The maps $\rho \mapsto q_{N,k}(\rho)$ are measurable, and the empirical measure of $\mathbf{q}_N(\rho)$ is exactly ρ.

On an extension of the Skorokhod probability space, take the Brownian motions, Poisson clocks, and partner choices used to construct the particle systems independently of (R_N, R). Let $\widetilde{\mu}_N(s)$ be the empirical measure at time s of the N-particle system started from $\mathbf{q}_N(R_N)$.

The generator is invariant under permutations of particle labels, and the empirical measure is itself permutation invariant. Hence the empirical evolution from a deterministic configuration depends only on its empirical measure. Since R_N has the same law as $\mu_N(0)$, it follows that

$$\mathcal{L}(R_N, \widetilde{\mu}_N(t)) = \mathcal{L}(\mu_N(0), \mu_N(t)). \tag{40}$$

Fix $t > 0$, and let d_E be a bounded metric generating the weak topology on E. On the almost-sure event in Equation (39), the deterministic sequence $R_N \in E_N$ converges weakly to R. Therefore Equation (36), applied after conditioning on (R_N, R), gives, for every $\varepsilon > 0$,

$$\mathbb{P}\left(d_E\big(\widetilde{\mu}_N(t), \Psi_t R\big) > \varepsilon \,\big|\, R_N, R\right) \longrightarrow 0 \qquad \text{almost surely.} \tag{41}$$

The conditional probabilities in Equation (41) are bounded by one, so dominated convergence yields

$$d_E\big(\widetilde{\mu}_N(t), \Psi_t R\big) \longrightarrow 0 \qquad \text{in probability.} \tag{42}$$

Together with Equation (39), this implies

$$\big(R_N, \widetilde{\mu}_N(t)\big) \longrightarrow \big(R, \Psi_t R\big) \qquad \text{in probability in } E \times E. \tag{43}$$

Consequently,

$$\mathcal{L}\big(R_N, \widetilde{\mu}_N(t)\big) \Longrightarrow \mathcal{L}\big(R, \Psi_t R\big). \tag{44}$$

Using Equation (40), the fact that $R \sim \Lambda$, and the fact that $\mu \sim \Lambda$, we conclude from Equation (44) that

$$\mathcal{L}(\mu_N(0), \mu_N(t)) \Longrightarrow \mathcal{L}(\mu, \Psi_t \mu)$$

for every fixed $t > 0$.

For $t = 0$, the result follows directly from Equation (37) and continuity of the diagonal map

$$E \longrightarrow E \times E, \qquad \rho \longmapsto (\rho, \rho),$$

since $\Psi_0 \rho = \rho$. This completes the proof. $\square$

B Proofs of Properties of F-KPP Solutions

Proof of Proposition 2. Let

$$\{X_1(t), \ldots, X_{N_t}(t)\}$$

be binary branching Brownian motion started from one particle at the origin, with branching rate one. Let F be the CDF of μ. By McKean's representation [5, formula (6), p. 325],

$$u(t,x) := \mathbb{E}\left[\prod_{i=1}^{N_t} F\big(x + X_i(t)\big)\right] \tag{45}$$

is the CDF of $\Psi_t\mu$. Write

$$P_tF(x) = \mathbb{E}[F(x+B_t)],$$

where B is a standard Brownian motion. We claim that, for every $x<y$,

$$e^{-t}\big(P_tF(y)-P_tF(x)\big) \le u(t,y)-u(t,x) \le e^{t}\big(P_tF(y)-P_tF(x)\big). \tag{46}$$

Restricting to the event that no branching occurs before time t, which has probability e^{-t}, gives

$$u(t,y)-u(t,x) \ge e^{-t}\mathbb{E}\big[F(y+B_t)-F(x+B_t)\big].$$

For the opposite inequality, use

$$0 \le \prod_{i=1}^{n} b_i - \prod_{i=1}^{n} a_i \le \sum_{i=1}^{n}(b_i-a_i), \qquad 0\le a_i\le b_i\le 1,$$

and then the standard many-to-one identity

$$\mathbb{E}\left[\sum_{i=1}^{N_t} g(X_i(t))\right] = e^t\mathbb{E}[g(B_t)].$$

If Φ denotes the standard Gaussian distribution function, then

$$P_tF(x) = \int_{\mathbb{R}} \Phi\left(\frac{x-z}{\sqrt{t}}\right)\mu(dz).$$

Consequently, P_tF is continuous and, whenever $x<y$,

$$P_tF(y)-P_tF(x) = \int_{\mathbb{R}} \left[\Phi\left(\frac{y-z}{\sqrt{t}}\right)-\Phi\left(\frac{x-z}{\sqrt{t}}\right)\right]\mu(dz) > 0.$$

It follows from Equation (46) that $u(t,\cdot)$ is continuous and strictly increasing.

It remains only to check its limits at infinity. Since $\prod_i a_i \le \sum_i a_i$ for $a_i\in[0,1]$, the many-to-one identity implies

$$0\le u(t,x) \le e^tP_tF(x) \longrightarrow 0 \qquad \text{as } x\to-\infty.$$

Similarly, using $1-\prod_i a_i \le \sum_i(1-a_i)$,

$$0 \le 1-u(t,x) \le e^t\mathbb{E}[1-F(x+B_t)] \longrightarrow 0 \qquad \text{as } x\to+\infty.$$

Thus $u(t,\cdot)$ is a continuous, strictly increasing distribution function. □

Proof of Proposition 3. Fix $c\in\mathbb{R}$ and $x_1\le x_2$, and suppose that

$$F_\mu(x_1) > F_\nu(x_1+c).$$

Since F_μ and F_ν are continuous and $\mu_N \Longrightarrow \mu$, $\nu_N \Longrightarrow \nu$, we have

$$F_{\mu_N}(x_1)\longrightarrow F_\mu(x_1), \qquad F_{\nu_N}(x_1+c)\longrightarrow F_\nu(x_1+c).$$

Hence, for all sufficiently large N,

$$F_{\mu_N}(x_1) > F_{\nu_N}(x_1+c).$$

Since $\mu_N \ge_{\mathrm{stp}} \nu_N$,

$$F_{\mu_N}(x_2) \ge F_{\nu_N}(x_2+c).$$

Passing to the limit gives

$$F_\mu(x_2) \ge F_\nu(x_2+c).$$

Thus $\mu \ge_{\mathrm{stp}} \nu$. □

The following lemma is Proposition 3.2 in [2].

Lemma 5. *Let $u^1(t,x)$ and $u^2(t,x)$ satisfy*

$$u_t = \frac{1}{2}u_{xx} + f(u)$$

where $f(0) = f(1) = 0$, $f(u) > 0$ for $0 < u < 1$ and $f'(0) = 1$ and $f'(u) \leq 1$ for $0 < u \leq 1$. Suppose if $x_1 < x_2$ then

$$u^2(0,x_1) > u^1(0,x_1) \implies u^2(0,x_2) \geq u^1(0,x_2).$$

Then for all t if $x_1 < x_2$,

$$u^2(t,x_1) \geq u^1(t,x_1) \implies u^2(t,x_2) \geq u^1(t,x_2).$$

Proof of Proposition 4. The case $t = 0$ is immediate. Fix $t > 0$ and $c \in \mathbb{R}$, and write

$$U_s(x) = F_{\Psi_s\mu}(x), \qquad V_s(x) = F_{\Psi_s\nu}(x).$$

Since U_s and V_s solve

$$\partial_s w = \frac{1}{2}\partial_{xx}w + w(w-1),$$

the functions

$$u(s,x) = 1 - U_s(x), \qquad v(s,x) = 1 - V_s(x+c)$$

solve the usual F-KPP equation

$$\partial_s w = \frac{1}{2}\partial_{xx}w + w(1-w).$$

Because $\mu \geq_{\text{stp}} \nu$, for every $x_1 \leq x_2$,

$$v(0,x_1) > u(0,x_1) \quad \implies \quad v(0,x_2) \geq u(0,x_2).$$

Hence Lemma 5 implies

$$v(t,x_1) > u(t,x_1) \quad \implies \quad v(t,x_2) \geq u(t,x_2).$$

Returning to U_t and V_t, this is exactly

$$U_t(x_1) > V_t(x_1+c) \quad \implies \quad U_t(x_2) \geq V_t(x_2+c).$$

Since $c \in \mathbb{R}$ was arbitrary,

$$\Psi_t\mu \geq_{\text{stp}} \Psi_t\nu.$$

The case of the centered flow follows from the translational invariance of steepness. □

Proof of Proposition 5. To see Equation (3), since $u(x,t) = U_{min}(x - \sqrt{2}t)$ satisfies Equation (2), we know

$$-\sqrt{2}U'_{\text{min}} = \frac{1}{2}U''_{\text{min}} + U_{\text{min}}(U_{\text{min}} - 1).$$

We know that

$$\int_{\mathbb{R}} U'_{\text{min}}(x)dx = U_{\text{min}}(\infty) - U_{\text{min}}(-\infty) = 1$$

and

$$\int_{\mathbb{R}} U''_{\text{min}}(x)dx = U'_{\text{min}}(\infty) - U'_{\text{min}}(-\infty) = 0,$$

so integrating over $\mathbb{R}$ gives

$$\sqrt{2} = \mathcal{S}(\pi_{\min}).$$

To show Equation (4), we will show something slightly more general. Suppose μ and ν are measures with continous strictly increasing CDFs F_μ and F_ν such that

$$F_\mu(0) = F_\nu(0) = 1/2$$

such that $\mu \geq_{stp} \nu$. Then we will show that $\mathcal{S}(\nu) \geq \mathcal{S}(\mu)$.

Given $x > 0$, take $h > 0$. Since $F_\mu(h) > F_\nu(0) = 1/2$, the definition of steepness gives that $F_\mu(x+h) \geq F_\nu(x)$. Taking $h \to 0$ using the continuity of F_μ gives $F_\mu(x) \geq F_\nu(x)$. Similarly for $x < 0$, we have $F_\mu(x) \leq F_\nu(x)$ (using the continuity of F_ν).

Since the function $x \mapsto x(1-x)$ is increasing for $x \in [0, 1/2]$ and decreasing for $x \in [1/2, 1]$, this means that for all $x \in \mathbb{R}$,

$$F_\mu(x)(1 - F_\mu(x)) \leq F_\nu(x)(1 - F_\nu(x))$$

which implies the result.

Under the same conditions on μ and ν, if $\mathcal{S}(\mu) = \mathcal{S}(\nu) < \infty$, then $F_\nu(x)(1-F_\nu(x)) - F_\mu(x)(1-F_\mu(x)) = 0$ for all x. The function $x \mapsto x(1-x)$ is strictly increasing on $[0, 1/2)$ and strictly decreasing on $(1/2, 1]$, so this implies that $F_\mu(x) = F_\nu(x)$ for all x where $F_\mu(x) \neq 1/2$ and $F_\nu(x) \neq 1/2$. Since $F_\mu(0) = F_\nu(0) = 1/2$ and both CDFs are strictly increasing, this means $F_\mu(x) = F_\nu(x)$ for all x. $\square$

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